transitive; and each number has a successor, including 4 since $4 + 1 \equiv 0 \pmod 5$. For this reason we need axioms 7 and 8. Axiom 8 rules out a loop like modular congruence since no number's successor can be 1. But even then a loop could start elsewhere, such as for the number list 1, $2 = S(1)$, $3 = S(2)$, $4 = S(3)$, $2 = S(4)$; no number has 1 as its successor but we still have a loop beginning with 2. Axiom 7 eliminates that possibility by ensuring that no two different numbers have the same successor ($2 = S(1)$ and $2 = S(4)$ would violate the axiom).

Finally, axiom 9 is needed to make sure that our set contains only the intended numbers. If $\mathbf{N} = \{1, 2, 3, 4, \ldots\}$ then **N** satisfies conditions (a) and (b) of axiom 9, and thus it must contain all natural numbers; no other object qualifies as a natural number.

One of the best ways to learn about a set of axioms is to see what structures do not satisfy the axioms.

Example 1

If we say that the set of natural numbers is $\{1, 2, 3, 4, 5\}$ with no successor for 5, which axiom is violated?

Solution:

Axiom 6 (Every natural number has a successor.)

When searching to determine which axiom is violated, axioms 1 and 6–9 are the prime candidates. Simply check these axioms one at a time to see whether or not they are satisfied. In example 1, our set was $\{1, 2, 3, 4, 5\}$. Is axiom 1 satisfied? Yes, since 1 is in our set. Is axiom 6 satisfied? No, since we have a number without a successor. For $a = 5$, $S(5)$ is supposed to be a natural number, and the axiom is violated by our set.

Example 2

If we say that the set of natural numbers is $\{1, 2, 3, 4, 5\}$ with $S(5) = 1$, which axiom is violated?

Solution:

Axiom 8 (1 is not the successor of any natural number.)

Searching through the axioms to determine the answer to example 2, we see that axiom 1 is satisfied since 1 is in our set. Skipping axioms 2–5, we see that axiom 6 is satisfied since every number has a successor. Axiom 7 is satisfied since no two numbers have the same successor. But axiom 8 is violated with $a = 5$, since $S(5) = 1$.