Section 1: Divisibility

The foundational topic for our study of number theory is *divisibility*. We shall begin with a definition.

> **Definition**
>
> Let a and n be integers. We say *a divides n*, written $a \mid n$, if there is an integer b such that $n = a \cdot b$. We then say *n is divisible by a*. Otherwise, we say a does not divide n, written $a \nmid n$.

Exploring the Definition

Does 3 divide 12? To answer the question, we need to see whether or not the definition is satisfied for a = 3 and $n = 12$ (the definition uses "*a* divides *n*," so "3 divides 12" would substitute 3 for a and 12 for n). We need to know if there is an integer b such that $n = a \cdot b$, that is, $12 = 3 \cdot b$. The answer is yes, since $12 = 3 \cdot 4$. We let $b = 4$, which is an integer (whole numbers and their negatives are integers). The definition is satisfied, and we may say $3 \mid 12$.

How do we proceed in general? In the definition, we are given two integers a and n, like 3 and 12 above. To determine that $a \mid n$, we need to find the integer b with the property $n = a \cdot b$. Above, b was 4. How do we find b? Since we want $n = a \cdot b$, we can use algebra to solve for b:

$$b = \frac{n}{a}.$$

In other words, we can find the needed b by division. For $a = 3$ and $n = 12$,

$$b = \frac{n}{a} = \frac{12}{3} = 4.$$

Since b is an integer, $a \mid n$; that is, $3 \mid 12$.

Now consider $a = 7$ and $n = 12$. We proceed by finding b:

$$b = \frac{n}{a} = \frac{12}{7} = 1.71 \quad \text{(see footnote[1])}$$

This time b is not an integer. Therefore we cannot say that 7 divides 12. It's true that $n = a \cdot b$, or $12 = 7 \cdot \frac{12}{7}$, but in order for 12 to be divisible by 7, $b = \frac{n}{a}$ must be an integer. This is when the "otherwise" clause in the definition kicks in. We thus say the following:

$$7 \nmid 12 \text{ since } \frac{12}{7} \text{ is not an integer.}$$

We are ready for some examples.

[1]Technically, $\frac{12}{7} \neq 1.71$. Many books write $\frac{12}{7} \approx 1.71$ to mean $\frac{12}{7}$ is approximately equal to 1.71. We shall avoid using $\approx$ for three reasons. First, most people use = anyway since it is easier to write. Second, it is nearly always clear in context what is meant. Finally, the difference between equal to and approximately equal to is often inconsequential.