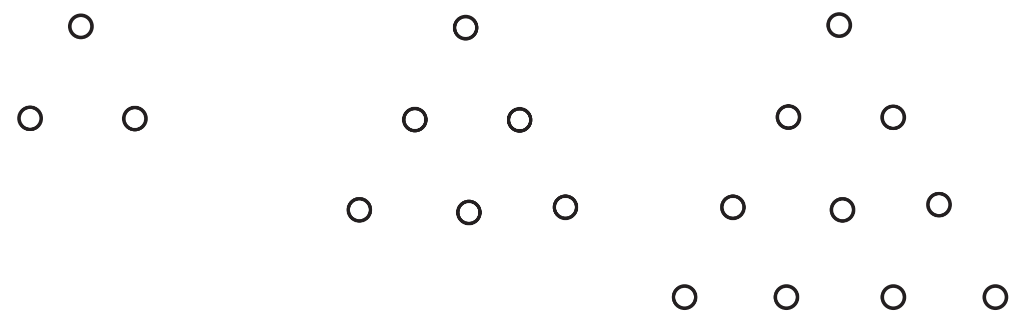
Section 5: Numbers of All Kinds

It is no secret that mathematicians love numbers. We study them, categorize them, compute them, and play with them. Certain properties of numbers can give rise to important applications, as we shall see in chapter 2. Other properties are studied for the sheer fun of it. That is the aim of this section.

Figurate Numbers

Have you ever taken cookies or candies and arranged them in a pattern before eating them? If your pattern formed a triangle, square, or similar geometric shape, then the number of cookies used is a *figurate number*.

Consider the number of cookies it takes to make an equilateral triangle:



$T_2 = 1 + 2 = 3$ cookies $T_3 = 1 + 2 + 3 = 6$ cookies $T_4 = 1 + 2 + 3 + 4 = 10$ cookies

The number of cookies necessary to form a triangle with n cookies on a side is thus $1 + 2 + \cdots + n$, which motivates the following definition.

Definition

The n^{th} *triangular number*, denoted T_n, is given by the sum of the first n positive integers:

$$T_n = 1 + 2 + \cdots + n.$$

The first ten triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45 and 55. Take a look again at the pictures of cookies. To go from a triangle with two cookies on each side to one with three on each side, we add a row of three cookies. To then obtain a triangle with four on each side, we add a row of four. It is apparent that if we have a triangle with $n - 1$ cookies on each side, we can make a triangle with n on each side by adding a row of n cookies. That justifies[10] the following *recursive formula*:

$$T_1 = 1,\ T_n = T_{n-1} + n \text{ for } n \geq 2.$$

A recursive formula gives a starting point for a sequence ($T_1 = 1$ above) and a rule for obtaining the next item in the sequence ($T_n = T_{n-1} + n$ for $n \geq 2$). To use the recursive formula, allow n to take on the values 2, 3, 4, . . .:

[10]This fact should also be apparent from the definition of triangular numbers.