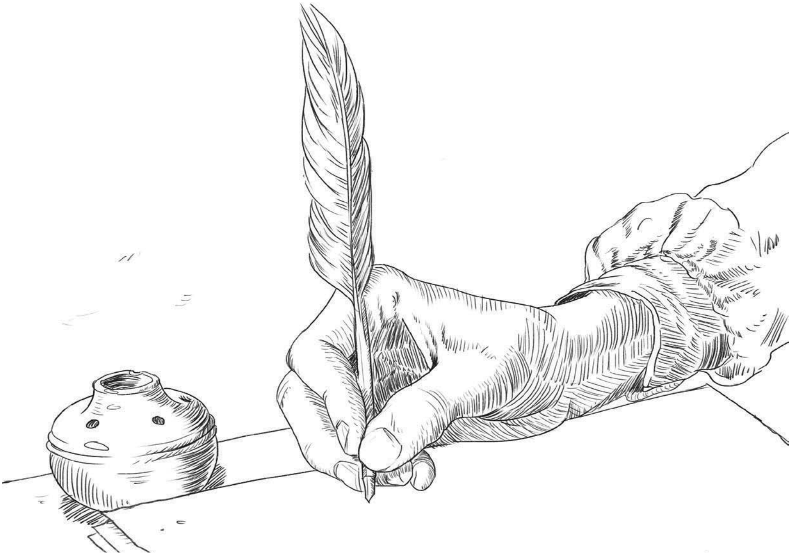
Mathematical Lives

Pascal and Fermat

The Probability Pen Pals



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Prologue

The Problem of the Points

The future was an unsolvable mystery, unknown and unknowable, forever hidden in an impenetrable fog that only lifted at a rate of one day per day, one hour per hour. Would it rain tomorrow? Would this year's harvest be a good one? Would you live to age 30? Only the local god or gods knew the answers. Some priests and mystics claimed to have special insight into what the gods knew, but everyday people weren't so blessed. For the everyday person, the only way to know the future was to live long enough to reach it.

At least, that was the way it seemed throughout most of human history. Astrologers, sages, and oracles all tried to predict the future. Poets and storytellers imagined what it would be like to have such power—but no one really did. The threads of fate were an invisible web that no one could untangle.

By the middle of the seventeenth century, however, things were starting to change. Mathematicians in Europe were beginning to think that, even if we couldn't know for certain what the future would bring, we could at least give ourselves an idea of what it *might* bring and how likely all the different possibilities were.

Today, we still can't know for certain whether it will rain tomorrow, but we can calculate the possibility that it *might* rain. We can't know how big the harvest will be, but we can calculate a range of possible sizes that it *might* be. And we can't know whether a person will reach age 30, but we can calculate how likely it is that he or she will. We can do that because of *probability*, the method of analyzing possible outcomes and calculating their chances of occurring.

And it all started with a question about gambling: *Suppose a group of men play a game of chance in which the first to reach a certain number of points wins all the money at stake. If the game is interrupted before anyone has enough points to win, how should the money be divided?*

In the early 1650s, a French aristocrat posed that question to Blaise Pascal, one of the most brilliant scientists and philosophers of the time. But even Pascal needed help. The idea of seeing the future—even seeing a *possible* future—was so alien that he needed to discuss it with someone else. So he contacted his countryman Pierre de Fermat, arguably the greatest "amateur" mathematician of all time. During the course of several months, the two men exchanged a series of letters that laid the foundation of what we know as *probability theory*.

In those letters, they changed the world.

Chapter 1

The Irresistible Subject

Blaise Pascal was born on June 19, 1623, in the city of Clermont (now Clermont-Ferrand), in the mountainous Auvergne province of south-central France. It was an age that Alexandre Dumas would later immortalize in his novel *The Three Musketeers*. Armand Jean du Plessis, Duke of Richelieu, had just become a cardinal in the Roman Catholic Church a year earlier. He was still a year away from becoming chief minister for King Louis XIII and going down in history as Cardinal Richelieu, one of the main characters Dumas wrote about. But there were no adventuring Musketeers around the Pascal home when Blaise was born. His father, Étienne, was a member of the French lower nobility and president of the region's *Cour des Aides*, the court that handled taxation matters for the king. His mother, Antoinette, was the daughter of a local merchant who had also been the Clermont sheriff.

Étienne and Antoinette married in 1616 and had their first child a year later. The baby girl, Anthonia, was baptized on Christmas Eve 1617 but died not long thereafter. A second daughter, Gilberte, was born in 1620, and she would be Blaise's devoted older sister throughout his life.

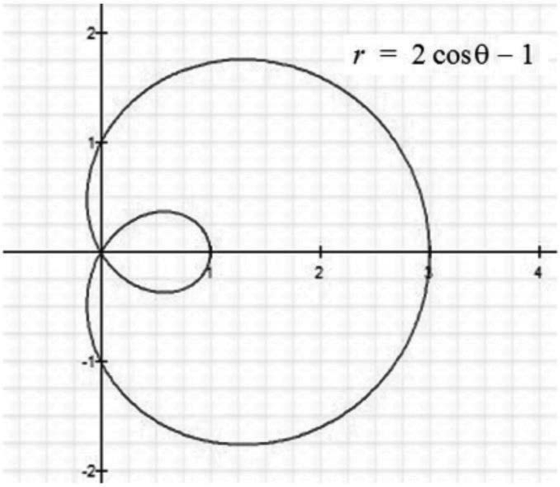
Sometime after Blaise's first birthday, he is said to have contracted a strange illness. In a memoir many years later, his niece Marguerite Périer described it as "*tombé en chartre*," a French expression meaning that he had become listless and that his body was wasting away, as if he had been a prisoner in a dungeon. Two other symptoms were even stranger. The sight of water sent him into a fit, as did the sight of his two parents together. Either parent alone could tend to him, and he gladly accepted their affection, but he would not tolerate having both near him at once. This condition went on for more than a year, and the family began to worry that he would die.

Marguerite went on to say that suspicion eventually fell upon one of the poor women in town who received alms from Antoinette every month. She was a witch, the townspeople claimed, and had cast a spell on Blaise. His parents tried to ignore the gossip, but the accusations grew louder as time passed and Blaise's condition grew worse. In the end, Étienne brought the woman to his office, hoping she would tell him something that could put the rumors to rest. Instead, much to his surprise, she confessed to casting a spell, saying that it was revenge for Étienne's refusal to represent her in a legal case. The curse was intended to be fatal, but it could be reversed through a ritual that included specially picked herbs and the sacrifice of a cat. The ritual was performed, and after a harrowing night during which Blaise appeared dead, the story concludes, he began to recover and was completely healed within three weeks.

Whatever Blaise's illness really was, it was the first sign of a fragile constitution that would plague him throughout his life. He was often in poor health, especially after he turned 18. By the time he wrote to Fermat in 1654, he was an invalid much of the time.

The Pascal household grew again in 1625 with the birth of Blaise's younger sister, Jacqueline. But tragedy struck when Antoinette died in 1626, leaving Étienne to raise the three children on his own. Blaise was only three years old. The family remained in Clermont until 1631, when Étienne sold his presidency of the *Cours des Aides*—a common practice at the time—and moved the children to Paris. Once there, he hired a governess and invested his money in a way that would provide for their living expenses, leaving him free to devote all his time and energy to educating his children himself.

Fortunately, Étienne was more than up to the task. He had earned a law degree in Paris in 1610, which made him well-versed in the classical studies of Latin and Greek. He was also an accomplished mathematician, especially in the field of geometry. Three years after the family moved to Paris, Cardinal Richelieu placed him on a committee evaluating a proposed method for sailors to find their longitude—how far east or west they were—while at sea. The committee spent five years on the problem before concluding that it wouldn't work without improved equipment that hadn't yet been invented. Much of that time was spent arguing with its inventor, who wanted the prize money that came with success.



Étienne even has a geometric shape named for him. The *Limaçon of Pascal*, named by his friend and colleague Gilles de Roberval, is a curve that looks like a snail shell (*limaçon* comes from *limax*, the Latin word for *snail*). In algebra, its equation is $(x^2 + y^2 - 2ax)^2 = b^2(x^2 + y^2)$, or $r = b + 2a \cos(\theta)$ in polar coordinates. You can draw one by tracing a point inside a circle that is rolling around the outside of an equally-sized circle (as you can do on a Spirograph, for example). Étienne began studying it in 1637. Among other things, it can be used to trisect an angle—that is, divide it into three equal parts—a geometry problem that has challenged mathematicians since the time of Euclid.

With that kind of ability, you might think that Étienne would begin teaching his children mathematics right away. But instead, he did the opposite.

Why? Because he wanted his children's education to be well-rounded, covering everything he knew, and he understood that Blaise's natural curiosity could make that goal difficult. When no one gave Blaise a good explanation for something that interested him, he launched into investigating it for himself. Gilberte later wrote that once, when Blaise was 11, he noticed that striking a china dish with a knife produced a loud noise and that the noise stopped when he put his hand on the dish. Wanting to understand why, he began a series of experiments and ended up writing a treatise on the nature of sound, his first scientific paper. Given these qualities, Étienne worried that once his son took up mathematics, he would devote himself to it so intently and so completely that he would wear himself out before mastering any other subjects. And so he forbade Blaise from studying any mathematics at all.

This, of course, was completely the wrong thing to do. Banning mathematics from the household only made Blaise more curious about it, and so he set about learning it on his own in secret. Gilberte claimed that Blaise drew circles and triangles in his room using burnt sticks and then worked on discovering the relationships between them. Other biographers tell a different story, claiming that Blaise found a copy of Euclid's *Elements*, the books on which all of geometry is founded, and read them in secret, finishing the first volume in a single afternoon. Whichever the case, Blaise took to the subject immediately, and within a short time was able to prove by himself the thirty-second proposition in the *Elements*, which states that the sum of the angles in a triangle is 180 degrees.

Once Étienne discovered what Blaise had been doing, he was so astonished and delighted by his son's ability that he lifted his ban on mathematics and began giving Blaise every book and explanation he could. He wanted to be sure that Blaise knew more than just the basic facts of mathematics. To Étienne, it was important that Blaise understand the methods and reasoning behind the facts because they are the key to making new discoveries.

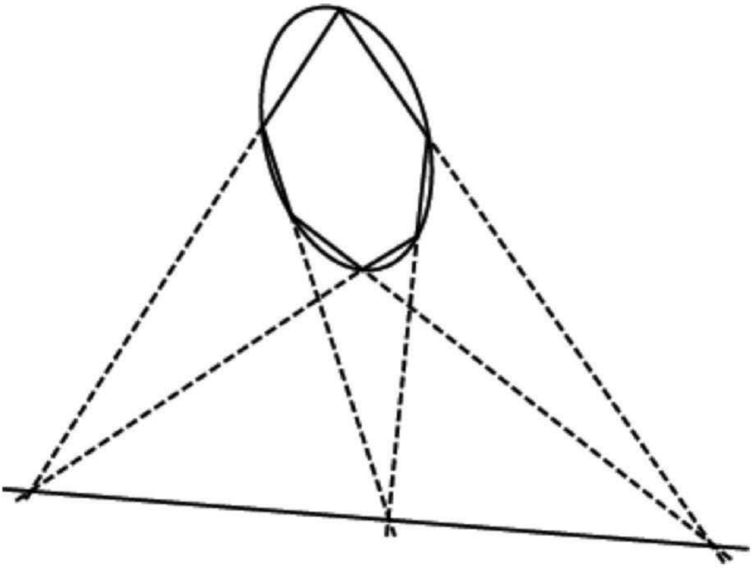
Étienne made one more important decision. France had many of the best mathematicians and scientists in Europe at that time, and in 1635, one of them, a priest named Marin Mersenne, formed the *Académie Parisienne*, an informal gathering of intellectuals. Not only did these men share and debate their theories and discoveries, but they also corresponded with other scientists in Europe and helped promote their work. Among the many great Renaissance minds they were in touch with was Galileo Galilei, who was under house arrest in Italy for claiming that the Earth orbited the Sun.

Étienne was one of the earliest members of Mersenne's *Académie*, and in 1637, he began bringing Blaise to the meetings with him. At 14 years old, Blaise only sat and listened quietly as the discussions went on around him, but within two years he was contributing discoveries of his own. In about 1639, he produced an essay on conic sections—the four shapes you get by passing a plane through a cone: a circle, an ellipse, a parabola, and a hyperbola. Blaise's work included the first piece of mathematics named for him, *Pascal's theorem*, which involved something he called the *Mystic Hexagram*.

The theorem applies to projective geometry, a different branch than the normal Euclidean geometry that most people study. However, it can be adapted for Euclidean geometry with a pair of extra rules. Both the theorem and its proof are surprisingly advanced for a teenager who had begun teaching himself the subject just a few years earlier.

Start with a conic section—any one of the four—and choose any six points on it. There are 60 different ways you can connect those points to form a hexagon—either a simple hexagon (in which the lines don't cross one another) or a self-crossing hexagon (in which, as the name suggests, the lines do cross one another). We'll use a simple hexagon as an example.

Next, extend two opposite sides until the lines cross each other. Do the same for a second pair of opposite sides, and then the remaining pair. Pascal's theorem states that those three intersection points will always form a straight line. All the lines taken together form the Mystic Hexagram.



It was an impressive start to what looked like a spectacular career of mathematical discoveries, especially since Blaise would continue to study and learn in the company of Mersenne's *Académie*. But it was not to be. Not long after Blaise wrote his essay, his father had to go into hiding, and the family soon found itself in a completely different part of France.

Chapter 2

The Pascaline

Throughout the 1630s, France was at war. In fact, the Thirty Years' War, as it would come to be called, had been raging across central Europe since 1618, and it would go on until 1648. It began as a conflict between Catholics and Protestants in the Holy Roman Empire, a region that included today's Germany, along with all or part of many other modern nations. But the nature of the war changed as France became more involved. France was a Catholic country, but it was also a longtime rival of the Catholic Habsburg dynasty that ruled both the Holy Roman Empire and Spain. In 1631, Cardinal Richelieu began giving financial support to Sweden, one of the leading forces on the Protestant side. After Sweden was defeated in 1634 and the Habsburgs neared victory, France entered the conflict directly, declaring war on Spain in 1635 and the Holy Roman Empire in 1636.

At first the fight went badly for the French. The Spanish Empire of the time included both Belgium and the Netherlands, and the French army tried attacking there in support of Belgian and Dutch Protestants. The Spanish armies pushed back, devastating several northern French provinces and even threatening Paris. At that point, the Spanish hesitated, unsure whether they could advance any farther. Their delay gave the French time to regroup, and the Spanish were forced to withdraw.

Wars do more than take lives and destroy territory; they also cost money—a lot of money. Much of the Thirty Years' War was fought by mercenaries, armies of men fighting for money, not for their homes or their monarchs. When they weren't paid, they looted and pillaged the countryside, taking whatever they could find in the nearby villages and towns. France had spent years paying to support Sweden's war effort. When France's own first campaign met with disaster, it needed more money to organize the defense of Paris, and once the Spanish had been driven back, it needed still more money to go on the offensive. By 1638, France was in serious financial trouble.

When Étienne Pascal moved his family to Paris, he invested his money in government bonds issued by the *Hôtel de Ville*, Paris's City Hall. The family had been living off the income that those bonds generated. But in 1638, Cardinal Richelieu, needing more money for the war effort, seized the *Hôtel de Ville*'s annual revenue for the crown, depriving Étienne and his fellow stockholders of their livelihoods.

It was not a popular decision. In March of that year, the angry stockholders converged on the office of the French Chancellor in a protest that threatened to become a small riot. When the Chancellor reported the incident to Richelieu, the Cardinal ordered the protest leaders arrested and locked in the Bastille, France's infamous prison. Étienne hadn't been at the protest in person, but he was known to be one of the group's leaders. So when the Cardinal issued his arrest warrant, Étienne's name was on it.

Fortunately for Étienne, he was not at the top of the Cardinal's list. Three of his compatriots were not so lucky. When Étienne learned of their arrests, he slipped out of Paris and made his way back to Auvergne. Several old friends took him in and kept him hidden for months. He remained out of sight through the rest of the year, coming home only when Jacqueline fell ill with smallpox in September. Once she recovered, he left Paris again and resumed his exile.

The New Year came and went with no change. Then, in February 1639, Richelieu decided on a whim that he wanted to see a tragic play called *Tyrannical Love*, performed by a cast of young girls. He instructed his niece, the Duchess d'Aiguillon, to organize the production, and one of the girls she asked to perform was Jacqueline Pascal. Even at age 13—and looking younger because of her small size—Jacqueline was known among Paris's nobility as a talented poet who had already written a poem and then recited it for King Louis's wife, Queen Anne. She was a natural choice for the production.

But Gilberte, who was running the household in her father's place, refused at first, saying, "The Cardinal has not been kind enough to us to make us take any pains to give him pleasure." The Duchess urged her to reconsider, suggesting that Jacqueline might use the occasion to ask for Étienne's pardon and even offering to use her influence with the Cardinal to help them. Gilberte's friends convinced her that such an offer was too good to pass up, and so Jacqueline found herself learning lines and preparing for the show.

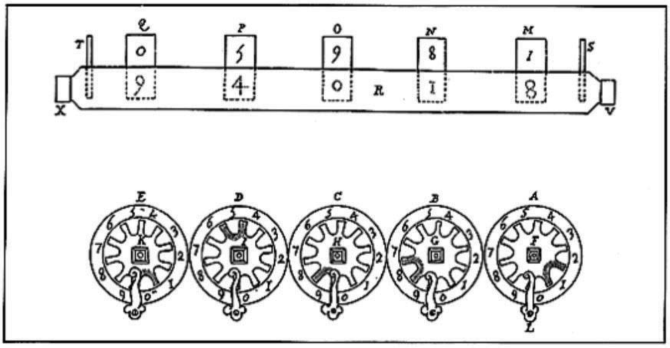
As Gilberte later recounted, the play was a success, with Jacqueline earning considerable praise for her performance. After the show was over, Jacqueline waited for the Duchess to present her to the Cardinal, but when she saw Richelieu getting up to leave, she went over to him alone. Supposedly the Cardinal was so moved by her pleadings that he agreed to pardon Étienne, and he invited the entire Pascal family to call on him at his castle on the outskirts of Paris. Not long after the visit, Richelieu gave Étienne a new job as the intendant in charge of finances for the province of Normandy.

Some modern historians scoff at Gilberte's story, skeptical that a shrewd and cunning ruler like Richelieu would be so easily persuaded by a young girl. However, the French treasury needed money, and there had been rebellions in Normandy over the high taxes being demanded. Richelieu could have used a chief tax collector who owed him a debt of gratitude. He may not have been swayed by Jacqueline's plea, but he certainly could have pretended he was. However it came about, the Pascal family soon found itself moving to Normandy's capital city, Rouen, about 80 miles up the River Seine, where they began a new chapter in their lives.

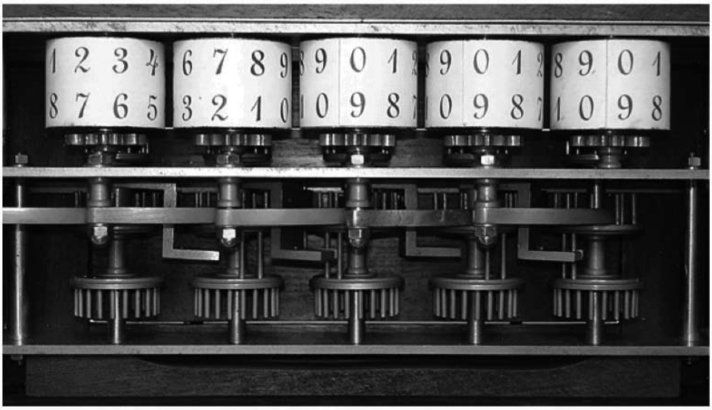
For Blaise, who had been working on his study of conic sections, the move meant that he could no longer attend Mersenne's *Académie*. For Étienne, being back at work meant that he could no longer devote himself to Blaise's education, although at that point Blaise hardly needed it. In fact, it wasn't long before Blaise began helping his father. Keeping track of the entire province's tax records meant long hours of grinding through tedious arithmetic. The calculations involved *deniers*, *sols*, and *livres*, the French currency of the time, with quantities in the millions. Soon Blaise was working on the calculations too, and so was Florin Périer, Blaise's distant cousin who would marry Gilberte in 1641. But even between the three of them, they couldn't keep up. That was when Blaise had a new idea: he could save them all time and effort by inventing a machine that would do the arithmetic for them—a mechanical calculator.

A few scientists had already tried their hands at inventing calculating machines. In 1617, Scotsman John Napier published the design for a mechanical device that fancifully came to be called "Napier's bones," which could be used for multiplication and division. A few years later, German astronomer Wilhelm Schickard invented what he called a "calculating clock," which combined "Napier's bones" with a tooth-wheeled system for addition and subtraction. Today we only have a description of Schickard's machine from letters he wrote to his friend Johannes Kepler. Only two calculating clocks were ever built, and modern historians examining the design drawings conclude that they would have needed additional wheels and springs to work properly.

Blaise's machine—called the *roue Paschaline* in French, or the *Pascaline* in English—used wheels, gears, and springs, like the calculating clock. The operator used a stylus to turn a set of wheels on the top panel, one for each digit. Wheels and gears inside the machine then turned a set of drums with two rows of digits printed on them. The lower row was the number being set, and the upper row was its *nine's complement*—that is, the result of subtracting the number you want from nine. (Thus a 9 had a 0 above it, an 8 had a 1 above it, and so on.) The numbers were displayed through windows in the top of the case, and a metal bar covered either of the two rows, depending on what kind of calculation was being done.



The Pascaline's biggest innovation was a mechanism for carrying numbers in addition. As each wheel was turned past the 9 digit, a pair of pins pushed a lever and ratchet that advanced the next wheel by one. The design allowed the wheels to move independently of one another and for multiple additions to be made in quick succession.



But the wheels only turned in a single direction, not one direction for addition and the other for subtraction. Instead, subtracting one number from another required turning the problem into an addition one using the nine's complement.

Here is how it worked: Start with any number *a*, and then find the next greatest power of 10, designated as 10^N. So if *a* was 53, then 10^N would be 100 (meaning N was equal to 2, the power of 10). And so the nine's complement $C_9^2(53)$ would be:

$$C_9^2(53) = 100 - 1 - 53 = 46$$

Or in general terms:

$$C_9^N(a) = 10^N - 1 - a$$

This means that the nine's complement of (a – b) would be:

$$C_9^N(a - b) = 10^N - 1 - (a - b)$$

And with a little bit of algebra, you can show that:

$$C_9^N(a - b) = C_9^N(a) + b$$

So to find (a – b) using the Pascaline, you had to enter the nine's complement of *a* and then add *b*. You would then find the nine's complement of your result to get the final answer. The second row of numbers on the drums made the task simpler because you didn't have to keep the nine's complements in your head as you worked.

$$\begin{array}{rccccc} & 4 & 0 & 7 & 3 & 2 \\ - & & 6 & 5 & 4 & 9 \\ \hline \end{array}$$

— — — — — — —

$$\begin{array}{rccccccl} & 5 & 9 & 2 & 6 & 7 & \longleftarrow & \text{nine's complement} \\ + & & 6 & 5 & 4 & 9 & & \\ \hline & 6 & 5 & 8 & 1 & 6 & & \end{array}$$

— — — — — — —

$$\begin{array}{rccccccl} & \mathbf{3} & \mathbf{4} & \mathbf{1} & \mathbf{8} & \mathbf{3} & \longleftarrow & \text{nine's complement of sum} \end{array}$$

The only way the Pascaline could perform multiplication or division was through repeated addition or subtraction operations. But thanks to the carrying mechanism, that process didn't take very long. The machine was still faster and more efficient than working out all the calculations by hand.

Blaise began designing the Pascaline in 1642, when he was 19. Over the next three years, he built 50 different prototype models, making changes and improvements each time. Some of them were too difficult to operate, or their mechanisms didn't move smoothly enough. Others broke down too easily when moved around or were affected too much by changes in the weather. He also made them out of different materials, some of wood, others of ebony and ivory, and still others of copper or other metal. But finally in 1645 he presented a working machine to the Chancellor of France and announced that another machine could be seen at the home of Gilles de Roberval, in case anyone thinking of buying one wanted a demonstration.

The Chancellor was delighted with the machine and worked on Blaise's behalf to secure its patent, which finally came in 1649. Blaise hoped to turn his invention into a full-time business, but in the end it proved to be too expensive for most people. The price ranged from 100 to 500 French *livres*, in an age when a person could live comfortably on 100 *livres* for an entire year. Blaise ended up selling only about a dozen machines.

The Pascaline was well-liked by the people who bought it. Queen Cristina of Sweden bought one, and Queen Marie Louise de Gonzague of Poland ordered two. But widespread demand for the machine never came, and eventually Blaise gave up on the project. Today, there are eight or nine Pascalines still in existence, either in private collections or on display in museums.

Between Étienne's work collecting taxes and Blaise's work on the Pascaline, life was always busy in the Pascal household. Gilberte had married and moved away, but Jacqueline remained at home, continuing to develop her poetic skills. Then, in 1646, Étienne slipped on a patch of ice and fell, dislocating his hip. The accident brought two new figures into the Pascal home who would start Blaise down the path toward the other major contributions he would make to the world.