Problemoids

Grade 4
Math Mentor

Revised Edition

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Introduction

Problemoids is a problem-solving program designed to meet the needs of intellectually gifted children, as well as those children who would benefit from an enriched mathematics program that is more advanced than that provided in the standard curriculum. It engages children in high-level thinking about challenging problems and provides a stimulating opportunity for them to develop and increase their repertoire of problem-solving strategies.

Problemoids Level 4 contains 36 problem sets with five problems in each set, equaling a total of 180 problems with solutions, which increase in difficulty as children progress through the book. The problems differ from those in a typical curriculum in that solving them requires children to use the full spectrum of thinking skills in Bloom's Taxonomy, with special emphasis on high-level skills such as analysis, synthesis, and evaluation. The topics covered in the program extend, enrich, and reinforce topics typically covered in a fourth-grade mathematics curriculum: sets, numbers and numeration, operations, algebra, geometry and measurement, and probability and statistics.

Strategy-directed hints accompany the main problem in each set to assist children in solving the problem and in learning problem-solving strategies. The solutions emphasize those strategies by illustrating how to use them to solve the problem.

GOALS

There are three main goals of the program:

1. Improving children's problem-solving abilities
2. Enriching children's mathematics abilities by teaching them problem-solving strategies
3. Assisting children in creating a "memory bank" of types of problems, solutions, and methods for arriving at solutions

Other objectives include:

1. Exercising children's high-level thinking skills by challenging them with problems that require the use of analysis, synthesis, and evaluation
2. Fostering children's sense of self-confidence in solving problems
3. Extending children's perseverance in attempting to solve challenging problems
4. Providing experiences that may serve to increase children's interest in mathematics

PROGRAM CONTENT

Each problem set in this book includes a main problem and four related problems called "looking-back questions." The strategies necessary to arrive at the solutions to the looking-back questions are related to those necessary to solve the main problem in the set.

The problems are designed to provide children with experience in choosing and using problem-solving strategies. The problems can't be solved by simply using an operation such as addition, subtraction, multiplication, or division; instead, children must use one or more of the following strategies, which have been adapted from the problem-solving strategies identified by the eminent mathematician George Polya:

Restate the problem in your own words.
Find the information given in the problem.
Identify what the problem is asking for.
Identify the conditions of the solution.
Use all given and implied information.
Solve a simpler problem.
Make a list or a chart.
Draw a diagram.
Use trial and error.
Solve part of the problem.
Search for a pattern.
Work backward.

As many as four of these strategy-directed hints accompany the main problem in each set. The first looking-back question in each set also includes a strategy-directed hint. Most of the other looking-back questions do not; instead, they provide opportunities for children to think of strategies themselves.

Generally, if there is more than one hint with a problem, the hints build on one another to help children arrive at a solution. Occasionally, however, a hint suggests a strategy that will enable children to use an alternate method of arriving at a solution than the hint(s) before it pointed toward. As such, not every hint in every question is necessary to work through the problem (although often they are). That said, there are multiple ways of solving most of the problems beyond those laid out through the hints, and as children progress through the book, they should be encouraged to become less reliant on the hints, deciding on their own which strategies might work to solve the problems in a way that works best for them.

SETTING THE TONE

The learning atmosphere is one of the most important aspects affecting the success of a problem-solving program, and it depends largely on the actions and the attitude of the instructor. A supportive atmosphere is essential, especially when it comes to children taking intellectual risks. Compliment children for playing hunches, and encourage others to be supportive. Your consistent enthusiasm and praise can make an enormous difference.

Take actions to ensure that children have plenty of successes during the early part of the program. As children experience success resulting from their persistent efforts, both their willingness to attempt problems and their perseverance in finding solutions will increase.

Encourage collaboration among groups of children, if possible, and allow them plenty of time to solve problems and respond to questions. Persistence is more important than how quickly someone solves a problem. Similarly, concentrate on the *process* of solving problems. Emphasize using and choosing problem-solving strategies, and give less attention to the answer. When reviewing solutions, don't ask, "What is the answer?"; ask, "How did you solve it?"

USING THE PROGRAM

Assign the Problems in Sequence

This program was designed with the idea that children will work the sets of problems in order and will work the main problem in each set first. After you review the solution to the main problem with the children, have them go on to solve the looking-back questions in order. Children will be better prepared to work each problem if they have worked the problems that come before it in the sequence. In most instances, problem sets in the book are related to one another.

Because the strategy for finding the solution to the first looking-back problem is similar to that used for the main problem or builds on that used for the main problem, the first looking-back question serves as reinforcement for that strategy. The other looking-back questions usually vary more from the main problem and become increasingly challenging. As a result, instructors can differentiate this program to enable children of all ability levels to achieve success in it, allowing struggling learners to skip problems that are too difficult and simply moving on to the next problem set.

There are two primary options that work well when implementing this program. The first is to choose one day a week for working on a problem set. On that day, introduce the main problem, allow the children to solve it, and then review the solution with them. After discussing the solution, allow the children to work at their own pace on the looking-back questions, assisting anyone who needs help. Children need not solve all of the looking-back questions in a set. The program is designed to provide substance for all students and depth for the ablest students. It is more appropriate to allow for individual levels of progress than to require everyone to finish all of the problems.

The second option is to introduce the main problem early in the week, allowing the children to solve it and then reviewing the solution with them. The children can then complete the looking-back questions on subsequent days of the week. You can review the solutions at the end of the week or as the children complete the problems.

Introducing the Problem

To introduce a problem, read the problem, or ask a child to read it. Make sure the children understand the problem statement. If necessary, you may want to suggest strategy-directed hints for understanding the problem, such as:

1. Lucky Combination

Zoey was telling her mother about her first day at school. After talking about her teacher and her classmates, Zoey said to her mother, "My locker number at school is a three-digit number. The product of the digits is 12. The sum of the digits is 9. The digit in the tens place is higher than the digit in the hundreds place and lower than the digit in the ones place. Do you know what my locker number is?"

Her mother smiled and replied, "I do."

Do you?

Hint 1

Identify the conditions of the solution. Does the problem tell whether or not the same digit can appear more than once in Zoey's number?

Because Zoey says, "The digit in the tens place is higher than the digit in the hundreds place and lower than the digit in the ones place," all of the digits must be different. Also, the ones digit is highest, the tens digit is second highest, and the hundreds digit is lowest.

Hint 2

Make a list of three-digit numbers using groups of three different one-digit numbers that sum to 9.

Because 0 is the lowest digit possible, if it is used, it must occupy the hundreds place. But if 0 were in the hundreds place, the number would be a two-digit number. Therefore, we will not use 0. Notice, too, that using 0 would make the product equal 0. Here are the possibilities:

$$1 + 2 + 6$$
$$1 + 3 + 5$$
$$2 + 3 + 4$$

Hint 3

Identify the conditions of the solution. Which group of numbers in your list has a product of 12?

Only 1 x 2 x 6 = 12. Therefore, Zoey's locker number must be a combination of the digits 1, 2, and 6.

Answer

The digits in the number are 1, 2, and 6, but because the digit in the tens place must be higher than the digit in the hundreds place and lower than the digit in the ones place, the locker number must be 126.

1. Lucky Combination: Looking-Back Questions

(A) Suppose the problem is the same, except the digits sum to 10, and their product is 30. What is the locker number?

(Hint: *Make a list* of three different one-digit numbers that sum to 10, and examine their products.)

Solution: We'll make another list, and here again, we will not use 0.

	Sum	**Product**
1 + 2 + 7	10	14
1 + 3 + 6	10	18
1 + 4 + 5	10	20
2 + 3 + 5	10	30

Only 2, 3, and 5 have a sum of 10 and a product of 30. Arranging them in the order that fits the conditions of the problem, we have 235.

(B) Suppose the problem is the same, except the digits sum to 12, and their product is 42. What is the locker number?

Solution: Again, we'll make a list, without using 0.

	Sum	**Product**
1 + 2 + 9	12	18
1 + 3 + 8	12	24
1 + 4 + 7	12	28
1 + 5 + 6	12	30
2 + 3 + 7	12	42
2 + 4 + 6	12	48
3 + 4 + 5	12	60

Only 2, 3, and 7 sum to 12 and have a product of 42. Arranging them in order to fit the conditions of the problem, we have 237.

(C) Suppose the problem is the same, except the digits sum to 20, and their product is 216? What is the locker number?

Solution: Here is our list for this problem:

	Sum	**Product**
3 + 8 + 9	20	216
4 + 7 + 9	20	252
5 + 6 + 9	20	270
5 + 7 + 8	20	280

Only 3, 8, and 9 sum to 20 and have a product of 216. Arranging them in order to fit the conditions of the problem, we have 389.

(D) Use the conditions in problem C, and suppose the sentence “The digit in the tens place is higher than the digit in the hundreds place and lower than the digit in the ones place” were changed to “All of the digits of my locker number are different.” What solution would you offer?

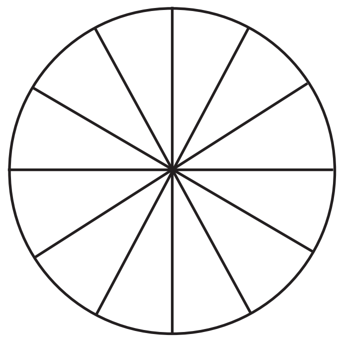
Solution: We can work only with the digits 3, 8, and 9. However, because the digits can be in any order, solutions of 389, 398, 839, 893, 938, and 983 are all possible. We can’t state the locker number with certainty.

16. Pizza Pieces

Carmine, the pizza chef and geometry whiz, is willing to cut your pizza almost any way you choose. The only things that he insists on are that each cut be straight and that each cut go from edge to edge all the way across the pie.

One day Scarlett asks Carmine to cut a round mushroom and pepperoni pizza six times.

"If I cut it the usual way, you'll get 12 pieces of pizza," Carmine says, pointing to a picture on the wall. "But you can get more than 12 pieces if I cut it another way."



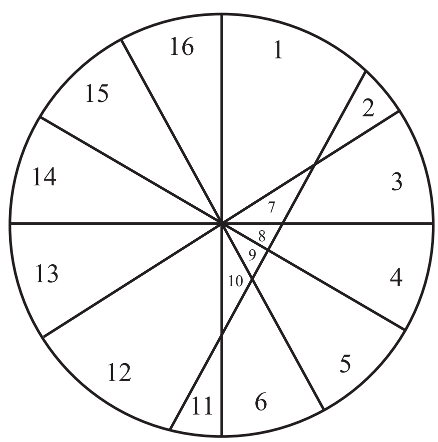
"I have a few friends coming over," Scarlett told him. "Cut it so that I get as many pieces as possible using six cuts."

How many pieces of pizza will Scarlett take home if Carmine makes the largest possible number of pieces using six straight cuts going from edge to edge all the way across the pie?

Hint 1

Use all given and implied information. Explore ways to increase the number of pieces using straight cuts all the way across the pie.

Notice that if just one of the six cuts is moved off center, we can get 16 pieces.

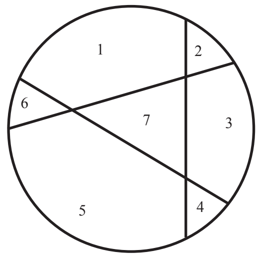


Hint 2

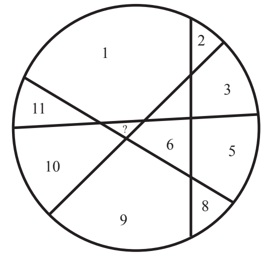
Solve a simpler problem. What is the largest number of pieces Scarlett could get if she told Carmine to cut the pizza three times? Four times?

If we make all of the cuts off-center, then we can cut the pizza as follows:

3 Cuts



4 Cuts



This gives us seven pieces with three cuts and 11 pieces with four cuts.

Hint 3

Make a chart, and *search for a pattern* in the chart.

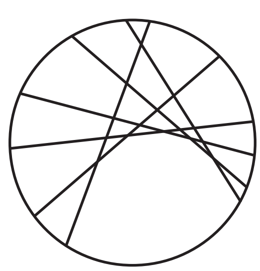
Here is the completed chart:

Number of Cuts	Largest Number of Pieces
1	2
2	4
3	7
4	11
5	16
6	22

Notice the pattern. We can calculate the largest number of pieces by adding the number of cuts we are using to the largest number of pieces for one cut less than we are using. Thus, two cuts equals 2 plus the largest number of pieces from one cut, which is 2, so $2 + 2 = 4$. Then, three cuts equals 3 plus the largest number of pieces from two cuts, which is 4, so $3 + 4 = 7$. The pattern continues with four cuts ($4 + 7 = 11$), five cuts ($5 + 11 = 16$), and six cuts ($6 + 16 = 22$). Therefore, the largest number of pieces with six cuts is 22.

Answer

Scarlett will get 22 pieces with six cuts.



16. Pizza Pieces: Looking-Back Questions

(A) How many pieces could Scarlett get with eight cuts?

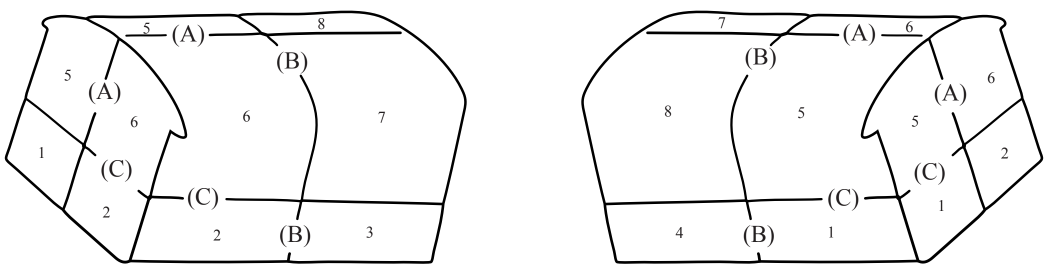
(Hint: *Search for a pattern*, and *use all given and implied information* to extend the pattern to eight cuts.)

Solution: Because Scarlett can get 22 pieces with six cuts, we can find that seven cuts can make 7 + 22 = 29 pieces, and eight cuts can make 8 + 29 = 37 pieces.

(B) Scarlett is going to cut a loaf of bread three times. What is the largest number of pieces of bread she can get?

(Hint: *Solve a simpler problem.* What is the largest number of pieces of bread that Scarlett can get with one cut? Two cuts?)

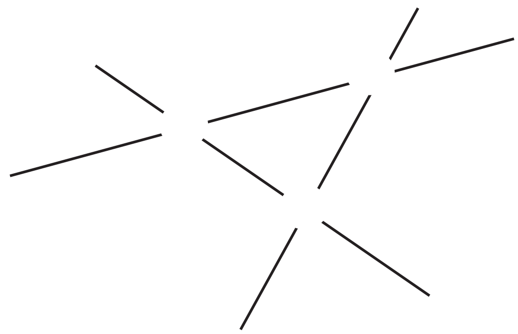
Solution: This problem is not the same as the pizza problem because the loaf of bread must be treated as a three-dimensional object. One cut produces two pieces of bread. Two cuts can produce four pieces of bread, as long as the second cut goes through both pieces produced by the first cut. Three cuts can produce eight pieces of bread if the third cut goes through each one of the pieces produced by the first two cuts.



(C) As the players on the all-star team are introduced and run onto the field, each one shakes hands with the players already on the field. If the all-star team has 10 players, how many handshakes are there altogether?

Solution: The first player shakes hands with nobody when he enters the field. The second player shakes hands with one player when he enters the field. The third player shakes hands with two people when he enters the field, the fourth with three people, the fifth with four people, and so on. An obvious *pattern* emerges. So by the time the tenth player has run onto the field, there must have been 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45 handshakes.

(D) A new computer game looks similar to an old game called Pick-Up Sticks. In the new game, the "sticks" are straight lines all the same length, and players must place 10 sticks in a pile. Wherever sticks touch each other, they cut each other in two. For example, the following image shows three sticks in a pile that together produce nine pieces:



What is the largest number of pieces that a player can produce with 10 sticks?

Solution: A player produces the highest number of pieces by placing each stick across every stick already in the pile. If we *solve some simpler problems*, we can see that one stick in a pile has only one piece, two sticks can produce four pieces, three sticks can produce nine pieces, and four sticks can produce 16 pieces. Next, we'll *search for a pattern* in the solutions to the simpler problems. In each solution, the highest number of pieces is the number of sticks times itself. For example, four sticks can produce 4 x 4 = 16 pieces. Using the pattern, we can predict that 10 sticks will produce 10 x 10 = 100 pieces.